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Conjectures for the first perimeter moment of directed animals

Y Le Borgne

LaBRI, Université de Bordeaux-CNRS, 351 cours de la Libération, 33405 Talence Cedex, France

E-mail: Yvan.LeBorgne@labri.fr

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Abstract
Using computer enumerations and a rational approximant method of series analysis, we conjecture an expression for the first perimeter moment of directed animals on the square lattice which are confined in a strip of a given width with open boundary conditions. When the width tends to infinity, the conjecture leads to an algebraic series for the first perimeter moment of directed animals on one-half of the square lattice, similar to Conway's earlier conjecture for the whole square lattice.

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In this paper, animals of source S are certain finite connected sets of sites of a directed square lattice that contain at least a subset S. The neighborhood of an animal is made up of sites that are not in the animal but are the target of a directed edge of the lattice coming from a site of the animal. The enumeration of these animals according to the number of sites, and the number of sites in the neighborhood, respectively called *area* and *perimeter* and encoded by the variables t and u, leads to the generating function $\boldsymbol{A}(S; t, u)$. This generating function is central in the context of site percolation on this directed lattice. Indeed, if sites of S are open and the other sites are open independently with probability p, the probability of non-percolation from any site of S is $\boldsymbol{A}(S; p, 1-p)/p^{|S|}$. Determining the bivariate generating function is an open problem at least as hard as computing this probability. The series $\left(\frac{\partial^k \boldsymbol{A}}{\partial u^k}(S; t, 1)\right)_{k\geqslant 0}$ appearing in the Taylor expansion at $u = 1$ of $\boldsymbol{A}(S; t, u)$ according to the variable u may be more tractable. Knowledge of all these series leads to $\boldsymbol{A}(S; t, u)$. Until the following section, we assume that $S = S_o$ consists of only a single given site. In this case, the first series in the Taylor expansion, the generating function $\boldsymbol{A}(S_o; t, 1)$ is now a well-known algebraic series [1, 9, 11], that is defined by equation (1) in this paper. Among the different proofs of this result or variants, some proofs begin with confining the sites of the animals to a strip of bounded width with open or cyclic boundary conditions and then the width tends to infinity to obtain the result on the whole square lattice [4, 12]. In the case of open boundary conditions, the generating function $\boldsymbol{A}_n(S_o; t, 1)$ of animals confined in a strip of width n is a rational function that can be written as $T_{n-1}(t/(1-t))/T_n(t/(1-t)) - 1$ where $(T_n(x))_{n\geqslant 0}$ are polynomials (defined after equation (4) in section 1 of this paper) related to Chebyshev's polynomials. Conway [7] formulated an algebraic equation, given as (2) of section 1 in this paper, satisfied by the second series of the Taylor expansion, $\frac{\partial \boldsymbol{A}}{\partial u}(S_o; t, 1)$, on the whole square lattice. This quantity is also called the first perimeter moment. This algebraic equation comes from an algebraic approximant of the first terms of the series, so this is not a proved equation. However, quoting Conway, 'since the approximant [is] generated from far fewer terms, it is exceedingly unlikely that [it is] incorrect'.

In section 1 we formulate a conjecture for the first perimeter moment, $\frac{\partial \boldsymbol{A}_n}{\partial u}(S_o; t, 1)$ for animals restricted to a strip of arbitrary width n with open boundary conditions. For a strip of width n not greater than 9, the transfer matrix approach allows us to compute exactly the first perimeter moment of animals confined in these strips in less than a minute on a computer algebra system. We can repeat this computation for a bit wider strips, but the time of computation becomes rapidly too long, and nine terms are sufficient to guess the denominators of these rational series and check this guess enough on thin strips to reasonably use it as a conjecture for all widths. For example in the case of width 5, using the change of variable $x = -t/(1+t)$ we obtain

$$\frac{\partial \boldsymbol{A}_5}{\partial u}(S_o; t, 1) = -\frac{(2 + 15x + 40x^2 + 45x^3 + 23x^4 + 8x^5 + 2x^6)x}{(1 + 5x + 6x^2 + x^3)^2} = \frac{P_5(x)}{T_5(-t/(1+t))^2}.$$

We observe that denominators in the first perimeter moments seem to be exactly the square of polynomials that are denominators of $\boldsymbol{A}_5(S_o; t; 1)$ the generating function according to area for the same width. This remark suggests considering whether the first perimeter moment is related to the derivative of $\boldsymbol{A}_n(S_o; t, 1)$ with respect to t, that is the generating function for animals with source S_o and with a site marked off. We did not find such relation. Stating a conjecture for numerators $(P_n(x))_{n\geqslant 0}$ requires to compute the first moment for wider strips. In section 2 we describe the alternation of computations and conjectures that we use to perform efficiently this computation. Our final conjecture, stated at the end of section 1, has for $n \geqslant 10$ the same status as Conway's conjecture: it is not proved but the approximants are generated by far fewer terms.

When the width tends to infinity, this conjecture leads to an algebraic equation similar to Conway's but for $\boldsymbol{A}_\infty(S_o; t, 1)$, the generating function of animals restricted to one-half of the square lattice which are called half-animals in this introduction. These animals appear in the combinatorial decomposition of animals of source S_o proposed in [1, 2] and re-presented in [5] where animals are described as a sequence of half-animals. This decomposition preserves the area and leads to

$$\boldsymbol{A}(S_o; t, 1) = \boldsymbol{A}_\infty(S_o; t, 1)(1 + \boldsymbol{A}(S_o; t, 1))$$

that explains in this case why the radical $\sqrt{(1+t)(1-3t)}$ appears simultaneously in both series $\boldsymbol{A}(S_o; t, 1)$ and $\boldsymbol{A}_\infty(S_o; t, 1)$. This radical comes from a quadratic equation related to a recursive decomposition of half-animals into up to two smaller half-animals. From this point of view, half-animals are more elementary than animals that is why we choose to fix the source on one boundary of the strip to obtain half-animals for infinite width and not in the middle of the strip, this choice giving animals for infinite width. All these decompositions are based on factorizations of heaps [14] in bijection with animals that imply non-uniform translations of the sites that do not preserve the perimeter. Thus we did not find a straightforward generalization of the decomposition of animals as a sequence of half-animals such that the perimeter of an animal can be deduced from its sequence of half-animals and in particular their

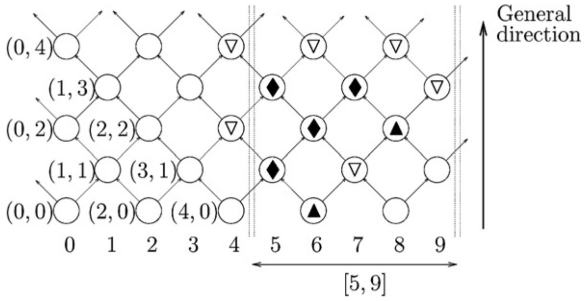

Figure 1. The labeling of the directed square lattice and an animal $A = \{\blacklozenge, \blacktriangle\} = \{(6,0),(5,1),(6,2),(8,2),(5,3),(7,5)\}$ of source $S = \{\blacktriangle\} = \{(6,0),(8,2)\}$ and neighborhood $N(A) = \{\triangledown\} = \{(7,1),(4,2),(9,3),(4,4),(6,4),(8,4)\}$. This animal is confined in the vertical strip defined by the interval [5, 9].

perimeters. However, Conway's conjecture (3) for animals and our conjecture (7) for half-animals also contain the same radical $\sqrt{(1+t)(1-3t)}$ that suggests the possible existence of such decomposition in the case of the first perimeter moment. Another interest of our conjecture is to approach a generating function similar to Conway's by a sequence of rational series whose numerators, respectively denominators, satisfy recurrences involving a finite number of numerators, respectively denominators. This sequence may be used in a proof by induction and play the same role in computation of the first perimeter as the ratio of consecutive Chebyshev's polynomials in the case of the enumeration of animals according to area. We hope that the publication of our conjecture will stimulate the research on this question like previous conjectures already have on directed animals [7, 8, 10, 13].

1. Definitions, some known results and the main conjecture

The *sites* of the two-dimensional directed square lattice are labeled by ordered pairs $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ such that $i + j$ is even. There is a *directed edge* $(i, j) \longrightarrow (k, l)$ from a site (i, j) to a site (k, l) if $(k, l) \in \{(i-1, j+1), (i+1, j+1)\}$. A *path* in this lattice is a sequence of sites $s_0, s_1, \ldots, s_n$ such that for $k = 0, \ldots, n-1$ there exists a directed edge $s_k \longrightarrow s_{k+1}$. All these and following definitions are illustrated in figure 1.

Let S be a finite set of sites of the square lattice. An *animal* A with source S is a finite set A such that for any site $a \in A$ there exists a path from a site of S to a made up of sites in A. A site $(i, j) \notin A$ is a *neighbor* in the *neighborhood* $N(A)$ of the animal A if there is an edge from a site in A to (i, j), that is, either $(i-1, j-1) \in A$ or $(i+1, j-1) \in A$. The set of animals with a given source S is denoted by $\mathcal{A}(S)$. The *area* $ar(A)$ of an animal A (with source S) is the cardinality of A and the *perimeter* $per(A)$ is the cardinality of its neighborhood $N(A)$. The distribution of these two parameters on $\mathcal{A}(S)$ defines the generating function $\boldsymbol{A}(S; t, u) = \sum_{A \in \mathcal{A}(S)} t^{ar(A)} u^{per(A)}$. We already know from the literature, see for example [1, 2, 9, 11] that

$$\boldsymbol{A}(S_o; t, 1) = \frac{1}{2}\left(\sqrt{\frac{1+t}{1-3t}} - 1\right) \tag{1}$$

where $S_o = \{(1, 1)\}$. Conway's conjecture (table 1 in [7]) is that $X(t) = \frac{\partial \boldsymbol{A}}{\partial u}(S_o; t, 1)$ satisfies the algebraic equation

$$tX(t)^2 + \frac{t^2+t+1}{1+t}X(t) + \frac{t(2-6t-5t^2+12t^3+13t^4+12t^5+9t^6)}{(1+t)^3(3t-1)^3} = 0. \tag{2}$$

It means

$$\frac{\partial \boldsymbol{A}}{\partial u}(S_o; t, 1) = \frac{1 - 3t + 2t^2 + t^3 - 3t^4}{2t(1+t)^2(1-3t)^2}\sqrt{(1+t)(1-3t)} - \frac{1+t+t^2}{2t(1+t)}. \tag{3}$$

An animal A is *confined in a (vertical) bounded strip* defined by an interval $[a, b] \subseteq \mathbb{Z}$ if any site (i, j) of A satisfies $i \in [a, b]$. The set of animals of source S included in the bounded strip is denoted $\mathcal{A}_{[a,b]}(S)$ and the corresponding generating function $\boldsymbol{A}_{[a,b]}(S; t, u)$. We remark that a neighbor site of an animal confined in a bounded strip may not be in this strip. With this notation we have $\boldsymbol{A}(S; t, u) = \boldsymbol{A}_{]-\infty,+\infty[}(S; t, u)$. To relieve the notations we use for any $k \geqslant 0$ and $n \geqslant 1$

$$\boldsymbol{A}_n^{(k)}(S; t, u) = \frac{\partial^k \boldsymbol{A}_{[1,n]}}{\partial u^k}(S; t, u).$$

We already know from the literature, see [5] citing [14], that for any $n \geqslant 1$

$$\boldsymbol{A}_n^{(0)}(S_o; t, 1) = \frac{T_{n-1}(-t/(1+t))}{T_n(-t/(1+t))} - 1, \tag{4}$$

where the polynomials $(T_n(x))_{n\geqslant -1}$, related to the Chebyshev polynomials, are defined by $T_{-1}(x) = 1 = T_0(x)$ and for $n \geqslant 1$, $T_n(x) = T_{n-1}(x) + xT_{n-2}(x)$. Since the substitution $t = -x/(1+x)$ appears frequently in this paper, we use the following unusual notation for the variables t and x: a function F of t where t is set to $-x/(1+x)$ is not denoted by $F(-x/(1+x))$ but more concisely by $F(x)$.

We remark that $\boldsymbol{A}_{+\infty}^{(1)}(S_o; t, 1)$ is the first perimeter moment of animals of source S_o lying in one-half of the square lattice. The following conjecture relies on computations detailed in the following section.

Conjecture 1. *For any $n \geqslant 1$,*

$$\boldsymbol{A}_n^{(1)}(S_o; t, 1) = \frac{P_n(x)}{T_n(x)^2} \tag{5}$$

where $x = -t/(1+t)$,

$$\begin{aligned}
P_1(x) &= -2x(1+x),\\
P_2(x) &= -x(2+3x),\\
P_3(x) &= -(1+x)x(2+5x+3x^2+x^3),\\
P_4(x) &= -(1+x)x(2+9x+11x^2+x^3)
\end{aligned}$$

and for $n \geqslant 5$,

$$P_n(x) = P_{n-1}(x) + 2x(1+x)P_{n-2}(x) + x^2P_{n-3}(x) - x^4P_{n-4}(x). \tag{6}$$

When n tends to infinity the series $\boldsymbol{A}_n^{(1)}(S_o; t, 1)$ converges to the power series

$$\boldsymbol{A}_{+\infty}^{(1)}(S_o; t, 1) = \frac{1 - 2t - t^2 - t^3}{2t^2(1+t)} - \frac{1 - 3t - t^2 - 2t^3 - 3t^4}{2t^2(1+t)(1-3t)}\left(\frac{1-3t}{1+t}\right)^{\frac{1}{2}}, \tag{7}$$

which satisfies the following algebraic equation where it is denoted by $X(t)$:

$$\frac{t(-2 + 3t + 9t^2 + 7t^3 + 5t^4 + 3t^5)}{(3t-1)(1+t)^2} + (t^3 + t^2 + 2t - 1)X(t) + t^2(1+t)X(t)^2 = 0. \tag{8}$$

For any power series $A(t)$, we denote by $[t^k]A(t)$ the coefficient of t^k in $A(t)$ and by $[t^{\leqslant k}]A(t)$ the polynomial $\sum_{i=0}^{k}([t^i]A(t))t^i$. As a consequence of conjecture 1, we can express given any $k \geqslant 1$ the first perimeter moment for animals on the half-plane whose area is exactly k:

$$[t^k]\boldsymbol{A}^{(1)}_{+\infty}(S_o; t, 1) = (-1)^k \sum_{j=1}^{k} \binom{k-1}{k-j} \left([x^j]\boldsymbol{A}^{(1)}_{+\infty}(S_o; x, 1)\right), \tag{9}$$

where $[x^1]\boldsymbol{A}^{(1)}_{+\infty}(S_o; x, 1) = -2$ and for $j \geqslant 2$

$$[x^j]\boldsymbol{A}^{(1)}_{+\infty}(S_o; x, 1) = \frac{1}{4}(-1)^j \binom{2j}{j} \frac{9j^4 + 37j^3 - 108j^2 + 92j - 72}{(2j-3)(2j-1)(j+1)(j+2)}. \tag{10}$$

The coefficient $[x^j]\boldsymbol{A}^{(1)}_{+\infty}(S_o; x, 1)$ is derived from equation (27) given in section 2.4 of this paper and the well-known expansion

$$\frac{1}{\sqrt{1-4x}} = \sum_{j \geqslant 0} \binom{2j}{j} x^j.$$

The fact that denominators in equation (27) are x^2, except for the radical $\sqrt{1+4x}$, simplifies the computation. Then the computation of the coefficient $[t^k]\boldsymbol{A}^{(1)}_{+\infty}(S_o; t, 1)$ is a straightforward translation of the substitution $x = -t/(1+t)$.

2. Computations which support the conjecture

As in [4, 12, 13], we use the well-known transfer-matrix approach to write systems of equations satisfied by the generating functions under study. Roughly speaking, we use the recursive deletion of the first row of an animal.

2.1. Computation to guess denominators and degree of numerators

A set of sites X is aligned (on the row k) if for all $(i, j) \in X$, $j = k$. The computation of the first terms of the generating functions are based on the following classical partition of animals with an aligned source S on the row k:

$$\mathcal{A}_{[1,n]}(S) = \bigcup_{T \subseteq N_{[1,n]}(S)} \left(\bigcup_{B \in \mathcal{A}_{[1,n]}(T)} \{S \cup B\} \right), \tag{11}$$

where $N_{[1,n]}(S) = \{(i, j) \in N(S) | 1 \leqslant i \leqslant n\}$ and so T is aligned on the row $k+1$. Knowledge of S and T is sufficient to determine the contribution of S to the area and perimeter. Moreover, for any aligned source S and interval $[1, n]$ we have

$$\boldsymbol{A}^{(0)}_n(S; t, u) = \boldsymbol{A}^{(0)}_n(\pi(S); t, u), \tag{12}$$

where π is the projection onto the two 'main' rows defined on the sites of the square lattice by $\pi((2i, j)) = (2i, 0)$ and $\pi((2i+1, j)) = (2i+1, 1)$. In terms of generating functions we obtain

$$\boldsymbol{A}^{(0)}_n(\pi(S); t, u) = \sum_{T \subseteq N_{[1,n]}(\pi(S))} t^{|\pi(S)|} u^{|N(\pi(S))| - |\pi(T)|} \boldsymbol{A}^{(0)}_n(\pi(T); t, u). \tag{13}$$

The set of projections $\pi(S)$ of aligned sources in the strip $[1, n]$ is finite and denoted Π_n. In this case, (13) is a finite set of linear equations whose unknowns are the $|\Pi_n|$ generating functions $\left(\boldsymbol{A}^{(0)}_n(S; t, u)\right)_{S \in \Pi_n}$ with the additional equation

$$\boldsymbol{A}^{(0)}_n(\emptyset; t, u) = 1.$$

A computer algebra system solves this system for a width n from 2 to 9 in less than a minute. Thus we obtain $\left(\boldsymbol{A}_n^{(0)}(S_o; t, u)\right)_{2\leqslant n\leqslant 9}$ which are rational functions in t with coefficients polynomials in u and a straightforward partial derivation leads to $\left(\boldsymbol{A}_n^{(1)}(S_o; t, 1)\right)_{2\leqslant n\leqslant 9}$ which are rational functions in t. Equation (4) suggests the reversible change of variable

$$x \equiv -\frac{t}{1+t}, \qquad \text{and conversely} \quad t \equiv -\frac{x}{1+x}. \tag{14}$$

This allows us to identify the denominators of $\left(\boldsymbol{A}_n^{(1)}(S_o; x, 1)\right)_{2\leqslant n\leqslant 9}$ and to note a strong regularity of the degree of numerators.

Conjecture 2. *For any $n \geqslant 3$, there is a polynomial $P_n(x)$ of degree $1+2\lceil n/2\rceil$ such that*

$$\boldsymbol{A}_n^{(1)}(S_o; x, 1) = \frac{P_n(x)}{T_n(x)^2}. \tag{15}$$

We remark that we also compute in this case the generating functions $\boldsymbol{A}_n(S; t, u)$ for any aligned source S but we do not use these results.

2.2. Computation of numerators

Assuming conjecture 2, the polynomial $P_n(x)$ can be computed from the first $1+2\lceil n/2\rceil$ non-zero terms of the expansion of $\boldsymbol{A}_n^{(1)}(S_o; x, 1)$:

$$P_n(x) = [x^{\leqslant 1+2\lceil n/2\rceil}]\left(T_n(x)^2\left([x^{\leqslant 1+2\lceil n/2\rceil}]\boldsymbol{A}_n^{(1)}(S_o; x, 1)\right)\right). \tag{16}$$

We deduce $[x^{\leqslant k}]\boldsymbol{A}_n^{(1)}(\pi(S); x, 1)$ from $[t^{\leqslant k}]\boldsymbol{A}_n^{(1)}(\pi(S); t, 1)$ since the substitution

$$t = -x/(1+x) = -x\left(\sum_{i\geqslant 0}(-x)^i\right) \tag{17}$$

maps a monomial t^k to a sum of monomials in x of degree at least k.

For any $j \in \mathbb{N}$, the series $\boldsymbol{A}_n^{(j)}(S; t, 1)$ is also the generating function of pairs (A, B) where A is an animal of source S and the set of sites $B \subseteq N(A)$ is a selection of j distinct neighbor sites on its neighborhood, according to the area of A. Using this remark, the system of equations (13) can be translated into the following recurrence over the coefficients:

$$[t^i]\boldsymbol{A}_n^{(j)}(\pi(S); t, 1) = \sum_{T\subseteq N_{[1,n]}(\pi(S))}\sum_{k=0}^{j}\binom{|N(\pi(S))| - |\pi(T)|}{k}[t^{i-|\pi(S)|}]\boldsymbol{A}_n^{(j-k)}(\pi(T); t, 1),$$

where k denotes the number of selected neighbors on $N(\pi(S))$. Like Conway in [7], we use dynamic programming to efficiently compute all $[t^k]\boldsymbol{A}_n^{(1)}(S_o; t, 1)$ for k from 1 to $1+2\lceil n/2\rceil$ using this recurrence. Only the coefficients of $\boldsymbol{A}_n^{(0)}(\pi(S); t, 1)$ and $\boldsymbol{A}_n^{(1)}(\pi(S); t, 1)$ are involved in this computation.

Assuming conjecture 2, we compute the numerators for n from 2 to 38. We also compute some additional terms of $\boldsymbol{A}_n^{(1)}(S_o; x, 1)$ to test this conjecture.

2.3. Computations on the sequence of numerators and denominators

To study the sequences of numerators, respectively denominators, we use an additional formal variable y to record the width of the strip and introduce the formal power series

$$N(x, y) \equiv \sum_{n\geqslant 2} P_n(x)y^n \qquad \text{and} \qquad D(x, y) \equiv \sum_{n\geqslant 2} T_n(x)^2 y^n.$$

In a previous work [3], the author conjectured by a haphazard inspection of the $P_n(x)$ a relation of the following form:

$$\alpha(x, y)N(x, y) + \beta(x, y)D(x, y) + \gamma(x, y) = 0,$$

where $\alpha(x, y)$, $\beta(x, y)$ and $\gamma(x, y)$ are polynomials in x and y. This relation allows us to deduce (8) but we present here only the following ansatz which seems not to be comparable to this relation. This ansatz was pointed to the author by Bousquet-Mélou: we check if $N(x, y)$ and $D(x, y)$ are rational functions in y with coefficients Laurent series in x. We computed enough terms of these series to use successfully the Padé approximation [6].

Conjecture 3. *The series of numerators and denominators are rational functions in y with coefficients Laurent series in x,*

$$N(x, y) = -\frac{xy(2 + 2x + xy + x^4y^2)}{(1 + xy)^2(1 - (1 + 2x)y + x^2y^2)} \tag{18}$$

$$D(x, y) = \frac{y((1 + x)^2 + x(1 + x - x^2)y - x^3y^2)}{(1 + xy)(1 - (1 + 2x)y + x^2y^2)}. \tag{19}$$

The sum of degrees with respect to y of numerators and denominators of $N(x, y)$ is 9 and we are able to compute the series expansion until y^{38} so, assuming conjecture 2, conjecture 3 is confirmed by the cancellation of roughly 30 terms. Recurrence (6) is a straightforward translation of the rational series $N(x, y)$.

2.4. Asymptotics of the ratios sequence

Proposition 1. *Conjecture 3 implies Conjecture 1*

Proof. We detail the study of the convergence of the ratios $(P_n(t)/T_n(t)^2)_{n\geqslant 2}$. We observe that the denominators of the two rational functions of (18) and (19) have the same roots with an additional multiplicity for one of them in $N(x, y)$. We denote the inverses of these roots by $\gamma_1(x) = x$, $\gamma_2(x) = 2x^2/(1 + 2x - \sqrt{1 + 4x})$ and $\gamma_3(x) = 2x^2/(1 + 2x + \sqrt{1 + 4x})$.

The partial fraction expansions of these rational functions lead to an expression of $[y^n]N(x, y) = P_n(x)$ and $[y^n]D(x, y) = T_n(x)^2$. For n large enough, we have

$$P_n(x) = \sum_{i=1}^{3} f_i(x)\gamma_i(x)^n + nf_4(x)\gamma_1(x)^n, \tag{20}$$

$$T_n(x)^2 = \sum_{i=1}^{3} g_i(x)\gamma_i(x)^n \tag{21}$$

where

$$f_2(x) = \frac{x(x^2 - 2x - 2)}{2(1 + 4x)} - \frac{x(2 + 8x + 5x^2 + 2x^3)}{2(1 + 4x)^2}\sqrt{1 + 4x}, \tag{22}$$

$$g_2(x) = \frac{1 + 4x + 2x^2}{2(1 + 4x)} + \frac{1 + 2x}{2(1 + 4x)}\sqrt{1 + 4x}. \tag{23}$$

We do not give explicitly the other Laurent series $(f_i(x))_{1\leqslant i\leqslant 4}$ and $(g_i(x))_{1\leqslant i\leqslant 3}$ as they are not related to the dominant terms that define the asymptotic behavior.

The *valuation* of a Laurent series $h(x) = \sum_{n=K}^{\infty} h_n x^n$, where $h_K \neq 0$, is K and denoted by $\mathrm{ldeg}(h)$. Since $\mathrm{ldeg}(\gamma_1(x)) = 1$, $\mathrm{ldeg}(\gamma_2(x)) = 0$ and $\mathrm{ldeg}(\gamma_3(x)) = 2$, we have

$$P_n(x) = f_2(x)\gamma_2(x)^n + \mathcal{O}(x^{n+k_P}), \tag{24}$$

where $k_P = \min(n\mathrm{ldeg}(\gamma_1) + \mathrm{ldeg}(f_1), n\mathrm{ldeg}(\gamma_3) + \mathrm{ldeg}(f_3), n\mathrm{ldeg}(\gamma_1) + \mathrm{ldeg}(f_4)) - n$ is an integer independent of n for n big enough, and

$$T_n(x)^2 = g_2(x)\gamma_2(x)^n + \mathcal{O}(x^{n+k_T}), \tag{25}$$

where $k_T = \min(n\mathrm{ldeg}(\gamma_1) + \mathrm{ldeg}(g_1), n\mathrm{ldeg}(\gamma_3) + \mathrm{ldeg}(g_3)) - n$. Since $\mathrm{ldeg}(\gamma_2(x)) = 0$, this power series admits an inverse which is a power series, so we can multiply by $1/\gamma_2(x)^n = \mathcal{O}(x^0)$ both terms in the following ratio:

$$\frac{P_n(x)}{T_n(x)^2} = \frac{f_2(x) + \mathcal{O}(x^{n+k_P})}{g_2(x) + \mathcal{O}(x^{n+k_T})} = \frac{f_2(x)}{g_2(x)} + \mathcal{O}(x^{n+k_{P/T}}) \tag{26}$$

where $k_{P/T} = \min(k_P, k_T) + \mathrm{ldeg}(g_2)$ as soon as $n > k_T + \mathrm{ldeg}(g_2)$.

Thus when n tends to infinity we have

$$\boldsymbol{A}^{(1)}_{+\infty}(S_o; x, 1) = \frac{f_2(x)}{g_2(x)} = \frac{1 + 5x + 6x^2 + 3x^3}{2x^2} + \frac{1 + 7x + 14x^2 + 13x^3 + 2x^4}{2x^2\sqrt{1+4x}}. \tag{27}$$

Using an argument similar to that involving (17) we deduce the value of $\boldsymbol{A}^{(1)}_{+\infty}(S_o; t, 1)$ in conjecture 1. □

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An implementation in Maple of the computations presented in this article is available from stacks.iop.org/JPhysA/41/335004.

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